Geometry of Higgs and Toda fields
on Riemann surfaces

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Abstract

We discuss geometrical aspects of Higgs systems and Toda field theory in the framework of the theory of vector bundles on Riemann surfaces of genus greater than one. We point out how Toda fields can be considered as equivalent to Higgs systems – a connection on a vector bundle E together with an $\mathrm{End}(E)$-valued one form both in the standard and in the Conformal Affine case. We discuss how variations of Hodge structures can arise in such a framework and determine holomorphic embeddings of Riemann surfaces into locally homogeneous spaces, thus giving hints to possible realizations of W_n-geometries.

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1. Introduction

The amount of research activity devoted to the study of conformal and integrable systems in two dimensions has reached a considerably high rate in the last years. In particular, much attention has been paid to Toda field theories and extended conformal (or W) symmetries, and the efforts done in this direction starting from the pioneering

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papers of Zamolodchikov [36], Gervais and Neveu [19] and Fateev and Lukyanov [16] have attained beautiful results both in the classical and in the quantum case.

On a more general level, as one of the most remarkable achievements can be also considered the bringing to light of the richness of the mathematical structure underlying such theories, and the deep relationships existing between *a priori* different field theoretical models.

In this paper we want to make a contribution in such a direction, namely to the study of some outcomes of the links between the Toda equations and the geometry of Higgs bundles or Higgs pairs, in the framework of the theory of (analytic) vector bundles (and connections thereof) on Riemann surfaces of generic genus.

A Higgs bundle is a system composed of a connection A on a vector bundle E over a Riemann surface Σ and a holomorphic endomorphism θ of E satisfying

$$F_A + [\theta, \theta^*] = 0. \tag{1.1}$$

Such structures were first introduced by Hitchin [26], in the framework of self-dual Yang Mills equations. The same author, in [27], proved their relevance showing that they consitute a remarkable example of an algebraically completely integrable hamiltonian system. Later on, a growing number of publications were devoted to the study of applications of Higgs pairs in the theory of harmonic bundles, local systems, uniformization and variations of Hodge Structures (see, e.g., [12,33,34]).

Our starting point is the zero-curvature representations of the Toda equations, (both in the standard and in the Conformal Affine cases) which in a suitable gauge (see also [21]) can be seen to be equivalent to Hitchin's equation (1.1) for the corresponding Higgs pair. Although it is not a difficult outcome of the Toda equations, *this is one of the central points of the paper.*

We can then proceed in two directions. At first we can adapt some nowadays fairly standard computations in Toda Field theory [9] to prove that the A_{n-1}-Toda connection, which is naturally defined on the direct sum $E = \bigoplus_{r=0}^{n-1} K^{-(n-1)/2+r}$ of powers of the canonical line bundle K on Σ, is mapped to an *analytic* flat connection on the full $(n-1)$th jet extension $V = J(K^{-(n-1)/2})$ of $K^{-(n-1)/2}$, whose degrees of freedom are parameterized by W_n-fields.

On the other hand, it is possible to decompose the Toda connection in a *metric* part plus a deformation α, which is simply the sum of the Higgs field and its metric conjugate. The structure equations for the Higgs pair translate into a harmonicity condition for the one-form α. This means that, associated to a Toda system, there is a natural harmonic twisted map f_H from Σ to a homogeneous manifold. Furthermore, the Toda connection (in the standard case), can be shown to satisfy the so-called Griffiths transversality conditions [23,33] and so defines a variation of a Hodge structure, a fact already noticed in [11] in the framework of $N = 2$ superconformal models and their integrable deformations. This entails that the map f_H is actually a *holomorphic embedding* of Σ into the quotient $\Gamma \setminus \mathbb{D}$ of a Griffiths period domain $\mathbb{D}$ by the monodromy group Γ.

Henceforth, through the theory of Higgs bundles we can associate to a Toda field (and, by the discussion above, to a W_n-algebra) a holomorphic map of Σ into a hermitean

manifold, a detour which can be also suggested by the fact that the construction of the Poisson commuting conserved quantities in [27] is reflected in the definition of W_n algebras via higher order Casimir invariants [7], and is under current investigation in the framework of Langlands–Drinfel'd correspondences (see, e.g., [18], and the references quoted therein).

It is natural thus to interpret such structures as a possible realizations of the W_n geometries, introduced in [35,20] (and more recently discussed in the framework of BRST symmetries in [37]). Although we do not perform here a thorough comparison between our framework and such results, we will make some comments about the relationship between our picture and the latest results of Gervais, Razumov and Saveliev [32,22] about W_n geometry and generalized Plücker embeddings associated to Toda systems (Section 6).

Let us sketch the plan of the paper. In Section 2 we collect some information about Higgs pairs and harmonic bundles from [26,12,34]. Then, in Section 3 we recall how the Toda equations can be seen as a zero-curvature condition, and describe the above mentioned equivalence between Toda fields and Higgs pairs. The extension to the Conformal Affine case is also discussed. We devote Section 4 to the description of the mapping between Toda bundles and W_n bundles on curves of arbitrary genus, pointing out their global aspects. In Section 5, after reviewing the basics of Griffiths theory of Variations of Hodge Structures, we show how the Toda equations fit into this framework and prove the holomorphicity of the embedding of Σ determined by the metric part of the Toda connection. We finally put our observations and comments in Section 6.

2. Higgs systems and harmonic bundles on a Riemann surface

Let Σ be a genus g Riemann surface with canonical bundle K.

Definition 1. A Higgs bundle over Σ is a pair (E, θ) with E a holomorphic vector bundle over Σ and $\theta \in H^0(\text{End } E \otimes K)$. A Higgs bundle is stable if Mumford's inequality

$$c_1(F)/\text{rk } F < c_1(E)/\text{rk } E$$

holds for every non-trivial θ-*invariant* subbundle $F \subset E$.

The generalization of the Narasimhan–Seshadri theorem holds in the following form [26,33]:

Theorem 2. *If (E, θ) is stable and $c_1(E) = 0$ there is a unique unitary connection ∇_H compatible with the holomorphic structure, such that*

$$F_H + [\theta, \theta^*] = 0. \tag{2.1}$$

The basic properties of Higgs systems we are going to use in the sequel are the following.

Given a connection ∇_H whose curvature equals the commutator $[\theta, \theta^*]$ for a holomorphic section $\theta \in H^0(\text{End } E \otimes K)$ then

$$\nabla = \nabla_H + \theta + \theta^* \tag{2.2}$$

is a *flat* $\text{GL}(n, \mathbb{C})$-connection, where $\rho(X) = -X^*$ is the anti-involution defining the compact real form of the group.

In [28] the holonomy of ∇ and ∇_H are characterized following the arguments below. Let $\mathfrak{g}^{\mathbb{C}}$ be a simple Lie algebra and let $\mathfrak{h} = \{h_1, e_1, f_1\}$ be a principal sl_2 subalgebra. Let $e_1, \ldots, e_k$ be highest weight vectors of the irreducible representations in which $\mathfrak{g}^{\mathbb{C}}$ is branched under $\mathfrak{h}$. Then there exists a Lie algebra involution σ of $\mathfrak{g}^{\mathbb{C}}$ sending $f_1 \rightarrow -f_1$ and $e_i \rightarrow -e_i, i = 1, \ldots, k$. The fixed point set of σ turns out to be the complexification of a maximal compact subalgebra of the split real form $\mathfrak{g}^r$ of $\mathfrak{g}^{\mathbb{C}}$.

Since σ commutes with ρ a careful application of Lie algebra properties along the lines of [29] proves that ∇_H has holonomy contained in the maximal compact subalgebra of $\mathfrak{g}^{\mathbb{C}}$, while the flat connection $\nabla = \nabla_H + \theta + \theta^*$ has holonomy contained in $\mathfrak{g}^r$.

The notion of Higgs system can be related to the one of *harmonic bundle* in a way we will briefly illustrate. A Higgs system (E, θ) such that (2.1) is satisfied clearly defines a pair (V, ∇), where V is the C^∞ bundle underlying E and ∇ is the flat connection (2.2). Let us now consider a complex rank n vector bundle V equipped with a flat connection ∇. As is well known, the introduction of an hermitean fibre metric H on V amounts to a reduction of the structure group from $GL(n, \mathbb{C})$ to $U(n)$, and allows for a splitting

$$\nabla = \nabla_H + \alpha, \tag{2.3}$$

where ∇_H is a unitary connection and α is a (C^∞) 1-form with values in (the self-adjoint part of) $\text{End}(V)$. Clearly, the connection ∇ being flat is equivalent to the following pair of equations [12]:

$$\nabla_H^2 + \tfrac{1}{2}[\alpha, \alpha] = 0, \tag{2.4}$$

$$\nabla_H \alpha = 0. \tag{2.5}$$

Definition 3 (Corlette, Donaldson, Simpson). We define the pair (V, ∇) to be *harmonic* or equivalently speak of a *harmonic metric* H if, under the splitting (2.3), we have

$$\nabla_H^* \alpha = 0, \tag{2.6}$$

where the adjoint is taken with respect to a given metric on Σ.

It is then a comparatively easy, but nontheless important remark that, since α is self-adjoint, we can decompose it as $\alpha = \theta + \theta^*$, thus showing that Eqs. (2.4), (2.5),

(2.6) are equivalent to Hitchin's self-duality equation (2.1), supplemented by $\bar{\partial}\,\theta = 0$ [3] [14,34].

The reason why a bundle or a metric satisfying (2.4), (2.5), (2.6) is called harmonic is to be understood in the following sense [12]. A metric H can be considered as a multivalued mapping $f_H : \Sigma \to GL(n,\mathbb{C})/U(n)$ or, in other words, as a section of a bundle over Σ whose standard fibre is the coset $GL(n,\mathbb{C})/U(n)$. Since ∇ is flat, V itself and all its associated bundles come from a representation of the fundamental group of Σ. The section f_H can in fact be regarded as a map from the universal cover of Σ, $f_H : \tilde{\Sigma} \to GL(n,\mathbb{C})/U(n)$, equivariant with respect to the action of $\pi_1(\Sigma)$. In other words we have the commutative diagram

$$
\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\ \ f_H\ \ } & GL(n,\mathbb{C})/U(n) \\[2mm]
\Big\downarrow & & \Big\downarrow \\[2mm]
\Sigma & \xrightarrow{\ \ f_H\ \ } & \Gamma \backslash GL(n,\mathbb{C})/U(n)
\end{array}
$$

Here $\pi_1(\Sigma)$ acts on $\tilde{\Sigma}$ as the group of Deck transformations and on $GL(n,\mathbb{C})$ via the holonomy representation Γ. It is well known that there exists a flat coordinate system for V in which ∇ is simply given by the exterior differential d. In these coordinates one has

$$\alpha = -\tfrac{1}{2} f_H^{-1}\, \mathrm{d}f_H\,, \tag{2.7}$$

which means that α can be identified with the differential of f_H, and therefore Eq. (2.6) implies that the map f_H is harmonic, according to the Eells–Sampson characterization of harmonic maps [15]. See [14,34] for details. We shall refer to the map f_H so obtained as the "classifying map" and by a slight abuse of language, a "harmonic bundle" will mean either the Higgs system (E,θ) satisfying (2.1) or the related C^∞ pair (V,∇).

Thus what we are going to consider in the sequel are harmonic bundles *plus* the additional structure given by the reduction of the structure group to a split real form [28].

Let us consider the bundle

$$E = \bigoplus_{r=0}^{n-1} K^{-(n-1)/2+r}\,. \tag{2.8}$$

Its determinant is trivial, therefore we can consider its structure group to be the semisimple group $G^{\mathbb{C}} = SL(n,\mathbb{C})$. According to [28], we take θ to be given by

$$\theta = \begin{pmatrix} 0 & 1 & & \cdots & 0 \\ u_2 & 0 & 1 & & 0 \\ u_3 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ u_n & \cdots & u_3 & u_2 & 0 \end{pmatrix}, \tag{2.9}$$

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[3] The $\bar{\partial}$-operator clearly comes from the (0,1) part of ∇_H.

where $u_r \in H^0(\Sigma, K^r)$ for $r = 2, \ldots, n$. It follows from [34], Lemmas 2.11 and 2.12, that the holonomy representation given by the pair (E, θ) is defined over $\mathbb{R}$ if and only if there exists a symmetric bilinear map $S : E \otimes E \rightarrow \mathcal{O}_\Sigma$ satisfying

$$S(\theta u, v) = S(u, \theta v)$$

for any two local sections u, v of E. This happens to be the case with the pair defined by (2.8) and (2.9), and with the map S given by the matrix [13]

$$S = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} . \tag{2.10}$$

Of course, this only tells us that the structure group, and hence the holonomy, is reduced to a real form of $G^{\mathbb{C}} = SL(n, \mathbb{C})$. Then Hitchin's analysis briefly reviewed earlier tells that this is in fact the *split* form $G^r = SL(n, \mathbb{R})$. It is of some interest to notice that the real form $SL(n, \mathbb{R})$ appears here in a rather disguised form, that is, the conjugation τ in $sl(n, \mathbb{C})$ which selects the split real form is concretely given by

$$\tau(\xi) = S \bar{\xi} S, \qquad \xi \in sl(n, \mathbb{C}) , \tag{2.11}$$

which nevertheless can be shown to be conjugate to the standard split real form given by $\xi \rightarrow \bar{\xi}$.

3. The Toda equations and their link with Higgs bundles

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and let us consider a Cartan–Weyl basis

$$\left[h_i, h_j \right] = 0 , \qquad \left[h_i, e_{\pm\alpha} \right] = \pm \alpha_i e_{\pm\alpha} , \qquad \left[e_\alpha, e_{-\alpha} \right] = \sum \alpha_i h_i .$$

A Toda field over a Riemann surface Σ is a field Φ taking values in the Cartan subalgebra of $\mathfrak{g}$ and satisfying the equations

$$\partial_z \partial_{\bar{z}} \Phi = \sum h_i \, e^{\alpha_i(\Phi)} . \tag{3.1}$$

It is a well known fact that Eqs. (3.1) can be obtained as the compatibility condition for a linear system [30]. Let us rederive this result in the framework of the theory of connections. Let us denote by Δ^s_+ (Δ^s_-) the set of all positive (negative) simple roots and by $\mathcal{E}_+$ $(\mathcal{E}_-)$ their sum, and define a $\mathfrak{g}$-valued local 1-form $A = A_z \, dz + A_{\bar{z}} \, d\bar{z}$ as

$$A_z = \tfrac{1}{2} \partial_z \Phi + \exp(\tfrac{1}{2} \operatorname{ad} \Phi) \cdot \mathcal{E}_+ , \tag{3.2}$$

$$A_{\bar{z}} = -\tfrac{1}{2} \partial_{\bar{z}} \Phi + \exp(-\tfrac{1}{2} \operatorname{ad} \Phi) \cdot \mathcal{E}_- . \tag{3.3}$$

The curvature F_A is a $(1, 1)$-form

$$F_{z \bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] .$$

Hence we get

$$F_{z\bar{z}} = -\partial_z\partial_{\bar{z}}\Phi + [\exp(\tfrac{1}{2}\operatorname{ad}\Phi)\cdot\mathcal{E}_+\,,\,\exp(-\tfrac{1}{2}\operatorname{ad}\Phi)\cdot\mathcal{E}_-]\,.$$

Since $[\Phi, E_\pm] = \pm\sum_{\alpha\in\Delta^r_+}\alpha(\Phi)e_{\pm\alpha}$, we have

$$\exp(\pm\tfrac{1}{2}\operatorname{ad}\Phi)\cdot\mathcal{E}_\pm = \sum_{\alpha\in\Delta^r_+}\exp\tfrac{1}{2}\alpha(\Phi)\,e_{\pm\alpha}$$

and so

$$F_{z\bar{z}} = -\partial_z\partial_{\bar{z}}\Phi + \sum_{\alpha,\beta\in\Delta^r_+}\exp(\tfrac{1}{2}\alpha(\Phi)+\beta(\Phi))\,[e_\alpha, e_\beta]$$

$$= -\partial_z\partial_{\bar{z}}\Phi + \sum h_i\exp(\alpha_i(\Phi))\,.$$

Let us now consider the gauge transformed connection under the element $g = \exp(\tfrac{1}{2}\Phi)$ [5]. We have

$$A_z^g = \partial_z\Phi + \mathcal{E}_+\,, \tag{3.4}$$

$$A_{\bar{z}}^g = \exp(-\operatorname{ad}\Phi)\cdot\mathcal{E}_-\,. \tag{3.5}$$

This form leads directly to the Higgs bundle picture. In fact, we can consider $\exp(\Phi)$ as an hermitean form on the fibers and split $D_A = \mathrm{d} + A$ as

$$D_H + \theta + \tilde{\theta}\,, \tag{3.6}$$

where D_H is the metric connection associated to $H = \exp(\Phi)$,

$$\theta = \mathcal{E}_+\,\mathrm{d}z \tag{3.7}$$

and $\tilde{\theta} = H^{-1}\mathcal{E}_-H$. Namely we have that

$$D_H{}' = \partial + \partial\Phi\,, \qquad D_H{}'' = \bar{\partial}\,.$$

Since the conjugation ρ acts as $\rho(h_i) = -h_i$, $\rho(e_i^+) = -e_i^-$, we get that $\tilde{\theta}$ is the metric adjoint endomorphism of θ. Hence, together with the obvious fact that $D_H{}''\theta = 0$, the zero-curvature equations in this gauge are

$$D_H{}^2 + [\theta, \theta^*] = 0\,, \tag{3.8}$$

thus showing that any solution to the Toda equations gives rise to a well defined solution of the Hitchin equations for the Higgs bundle. We follow [21] and call *Toda-gauge* the one where the connection takes the form (3.4), (3.5).

The endomorphism θ constructed above correspond exactly to the one given by (2.9) if we set all the u_r's to zero, while the vector bundle E is given by (2.8), where $n-1$ is the rank of the Lie algebra, i.e., $\mathfrak{g}^{\mathbb{C}} = A_{n-1}$. The metric H is given by a diagonal matrix whose entries $h_r = e^{\varphi_r}$, $r = 1,\ldots,n$, are themselves metrics on the factors $K^{-(n-1)/2+r-1}$ appearing in (2.8). This completely fixes the transformation law of the

fields φ_r, and one can check it coincides with the well-known conformal transformation properties of the Toda fields [5].

We wish now to extend the correspondence between Toda-like models and Higgs systems to the so-called Conformal Affine Toda models [6]. To this purpose, we retain the same form for the metric H but modify the endomorphism θ to

$$\theta = \mathcal{E}_+ + u\,e_{-\psi}\,, \tag{3.9}$$

where ψ is the longest root and $e_{\pm\psi}$ are the positive and negative root vectors. It follows that θ^* will be given by

$$\theta^* = \exp(-\operatorname{ad}\Phi)\cdot(\mathcal{E}_- + \bar{u}\,e_\psi) \tag{3.10}$$

(notice that $\mathcal{E}_- + \bar{u}\,e_\psi = -\rho(\mathcal{E}_+ + u\,e_{-\psi})$). Since ψ is the longest root, the element $e_{-\psi}$ has degree $-k \equiv -\operatorname{rank}\mathfrak{g}$ with respect to the action of the principal sl_2 subalgebra $\mathfrak{h}$ [29] and therefore we must have $u \in H^0(\Sigma, K^{k+1})$ [28], which in the A_{n-1}-case, for instance, means that u is a $(n,0)$-weight differential, i.e. a section of $(T_\Sigma^*)^{\otimes n} \equiv K^n$.

To clarify the procedure, let us decompose $A = B + \eta$, where η is the 1-form

$$\begin{aligned}\eta &= u\,e_{-\psi}\,\mathrm{d}z + \bar{u}\,\mathrm{e}^{-\operatorname{ad}\Phi}\,e_\psi\,\mathrm{d}\bar{z}\\&= u\,e_{-\psi}\,\mathrm{d}z + \bar{u}\,\mathrm{e}^{-\psi(\Phi)}\,e_\psi\,\mathrm{d}\bar{z}\,,\end{aligned}$$

in such a way that B is the connection associated to the standard Toda equations. Using the following fairly standard relation:

$$F_A = F_B + \tfrac{1}{2}[\eta,\eta] + D_B\eta$$

and $[\mathcal{E}_+, e_\psi] = [\mathrm{e}^{-\operatorname{ad}\Phi}\mathcal{E}_-, e_{-\psi}] = 0$, because ψ is the longest root, we calculate

$$\begin{aligned}\tfrac{1}{2}[\eta,\eta] &= -|u|^2\,\mathrm{e}^{-\psi(\Phi)}\,[e_\psi, e_{-\psi}]\,\mathrm{d}z \wedge \mathrm{d}\bar{z}\,,\\D_B\eta &= \left(\overline{\partial_{\bar{z}}u}\,\mathrm{e}^{-\psi(\Phi)}\,e_\psi - \partial_{\bar{z}}u\,e_{-\psi}\right)\,\mathrm{d}z \wedge \mathrm{d}\bar{z}\,,\end{aligned}$$

so that looking at the generators, we see that the equation $F_A = 0$ implies

$$F_B + \tfrac{1}{2}[\eta,\eta] = 0\,, \qquad D_B\eta = 0$$

separately, which gives

$$\partial_z\partial_{\bar{z}}\Phi = \sum h_i\,\mathrm{e}^{\alpha_i(\Phi)} - |u|^2\,\mathrm{e}^{-\psi(\Phi)}\,h_\psi\,, \tag{3.11}$$

$$\bar{\partial}u = 0\,, \tag{3.12}$$

where $h_\psi = [e_\psi, e_{-\psi}]$. Now, let us locally put $|u|^2 = \mathrm{e}^{2\eta}$ so that $\bar{\partial}u = 0$ yields $\partial\bar{\partial}\eta = 0$.

Then Eqs. (3.11) and (3.12) read

$$\partial_z \partial_{\bar{z}} \Phi = \sum h_i \, e^{\alpha_i(\Phi)} - e^{2\eta - \psi(\Phi)} \, h_\psi \, , \qquad (3.13)$$

$$\partial_z \partial_{\bar{z}} \eta = 0 \, , \qquad (3.14)$$

which essentially coincide with the equations defining the "Conformal Affine Toda" model [6,2]. Actually, in our formulation the field associated to the extra central generator in the affine algebra is missing. However this is not a serious problem, as the dynamics of this missing field is completely fixed by the other ones, whose equations of motion are correctly reproduced. We wish to stress that the conformal invariance of Eqs. (3.13), (3.14) naturally arises in the present setting, as they are interpreted as the integrability condition for a connection on a globally well-defined vector bundle.

We can summarize the results of this section in the following

Proposition 4. *A solution of the standard Toda equations (3.1) gives rise to a well defined solution of Hitchin's equation (2.1), whose underlying Higgs system is given by (3.7) above. The same statement applies to the Conformal Affine Toda equations (3.13), (3.14), with (3.7) replaced by (3.9).*

Let us remark that the above set-up allows us to interpret the limit from Conformal Affine Toda to standard Toda (see [6]) in a nice geometrical fashion. In fact, according to [28], we can regard the former as a "deformation" of the standard Toda model related to a deformation of the associated Higgs bundle.

4. Toda systems and W_n algebras

In this section we will make explicit some aspects of the relations between Toda equations and W_n algebras [19,9] through the theory of connections. It has already been established (see, e.g., [16,8]) that classical W_n algebras can be associated to the Drinfel'd–Sokolov reduction of a first order matrix differential operator (i.e. a connection) with respect to a parabolic subgroup. Here we show how the field content of such a theory can be obtained starting from a solution of the Toda equations in the case of systems defined on a higher genus Riemann surface. We shall confine ourselves to the case of standard A_{n-1}-Toda equations.

Let us recall that since a Riemann surface has complex dimension 1, the $(0, 1)$ part of any connection ∇ is automatically integrable, thus giving a holomorphic structure to the complex vector bundle supporting it [4]. In this holomorphic frame one has $\nabla'' = \bar{\partial}$. It will be convenient to refer to the holomorphic bundle so obtained as the *holomorphic bundle defined by (or associated to)* ∇''. Then, if the connection happens to be flat, its local $(1, 0)$-forms will be holomorphic in the holomorphic gauge. For a complex vector bundle to admit a holomorphic connection is a completely non trivial fact [3,25], since Weil's theorem states that such a bundle must be a direct sum of indecomposable flat bundles.

We want now to construct the holomorphic bundle (in the above sense) associated to the basic Higgs bundle (2.8), equipped with the connection

$$
D' = \partial +
\begin{pmatrix}
\partial\varphi_1 & 1 & & & \\
& \partial\varphi_2 & 1 & & \\
& & \ddots & & \\
& & & \partial\varphi_{n-1} & 1 \\
& & & & \partial\varphi_n
\end{pmatrix},
$$

$$
D'' = \bar{\partial} +
\begin{pmatrix}
0 & & & & \\
e^{\varphi_1-\varphi_2} & 0 & & & \\
& e^{\varphi_2-\varphi_3} & 0 & & \\
& & \ddots & \ddots & \\
& & & e^{\varphi_{n-1}-\varphi_n} & 0
\end{pmatrix}
\tag{4.1}
$$

(here $\sum_{i=1}^{n} \varphi_i = 0$), namely we want to prove the following

Theorem 5. *The holomorphic vector bundle V defined by the flat Toda connection $D = D_H + \theta + \theta^*$ is the vector bundle of $(n-1)$-jets of sections of $K^{-(n-1)/2}$. The holomorphic connection ∇, which is the image of the Toda connection D, has the standard W (or Drinfel'd–Sokolov) form:*

$$
\nabla' = \partial +
\begin{pmatrix}
0 & 1 & & & \\
& 0 & 1 & & \\
& & & \vdots & \\
& & & 0 & 1 \\
w_n & w_{n-1} & \cdots & w_2 & 0
\end{pmatrix}, \qquad \nabla'' = \bar{\partial},
\tag{4.2}
$$

with $\bar{\partial} w_i = 0$, $i = 2, \ldots n$.

Remark 6. We wish to stress the following point. The vector bundle E is a holomorphic bundle equipped with the C^∞ connection D. Its holomorphic structure is given simply by $D''_H = \bar{\partial}$. On the other hand V is the holomorphic vector bundle defined by the holomorphic structure D''. Thus the two vector bundles E and V are *holomorphically distinct*, although they are smoothly (i.e. C^∞) equivalent.

Remark 7. We point out that the previous theorem explicitly constructs the holomorphic vector bundle defined by D'', where D is the Toda connection, and identifies it with a concrete one (a holomorphic jet bundle).

The proof of theorem (5) will be divided into steps, or propositions, which are also of independent interest. In more detail, we want to show that the vector bundle E, associated in a suitable covering $\{\mathcal{U}_\alpha\}$ of Σ with the $SL(n, \mathbb{C})$-cocycle

$$
\mathcal{E}_{\alpha\beta} = \begin{pmatrix} k_{\alpha\beta}^{(n-1)/2} & & & & \\ & k_{\alpha\beta}^{(n-1)/2-1} & & & \\ & & \ddots & & \\ & & & k_{\alpha\beta}^{-(n-1)/2+1} & \\ & & & & k_{\alpha\beta}^{-(n-1)/2} \end{pmatrix} \tag{4.3}
$$

equipped with the connection D is C^∞-equivalent to the bundle V of $(n-1)$-jets of sections of $K^{-(n-1)/2}$, equipped with the connection ∇. We recall that the transition functions $\mathcal{V}_{\alpha\beta}$ of V can be gotten by expanding the relation $\partial_\alpha^l \xi_\alpha = (k_{\alpha\beta}^{-1}\partial_\beta)^l(k_{\alpha\beta}^{(n-1)/2}\xi_\beta)$, ξ_α being a local section[4] of $K^{-(n-1)/2}$, and $k_{\alpha\beta} = \partial z_\alpha/\partial z_\beta$.

We first discuss the transformation of D into ∇. The following proposition is well-known, see [30,5,21], and it is implicitly contained in the calculations of [1]. However, for the reader's convenience and to avoid unpleasant gaps in the arguments proving Theorem 5, we feel it worth stating it here and sketching its proof. It is completely local in character.

Proposition 8. *There is a sequence G_k, $k = 1,\ldots,n-1$, of gauge transformations, taking their values in the lower nilpotent part of $SL(n,\mathbb{C})$, such that the connection D is mapped by $G = \prod_{k=1}^{n-1} G_k$ into ∇.*

Proof. It is a straightforward albeit long calculation, so that we only illustrate the strategy and the first step. The idea is to show that a column at a time can be operated on using (lower) nilpotent abelian subalgebras of $sl(n,\mathbb{C})$. If E_{ij} is the standard matrix with 1 in the ij entry and zero elsewhere, we put

$$
G = \prod_{k=1}^{n-1} G_k, \tag{4.4}
$$

where

$$
G_k = \exp \sum_{j=k+1}^{n} g_j^{(k)} E_{jk} = \prod_{j=k+1}^{n} \exp g_j^{(k)} E_{jk}
$$

and recursively determine the $g_j^{(k)}$'s.

For instance, at step #1 we have to consider $G_1 = \exp\sum_{j=2}^{n} g_j^{(1)} E_{j1}$,

$$
\left(D''\right)^{G_1} = (\bar{\partial} g_2^{(1)} + e^{\alpha_1(\Phi)}) E_{21} \\
+ \sum_{i=2}^{n-1} e^{\alpha_i(\Phi)} E_{i+1\,i} + \sum_{i=2}^{n-1} (\bar{\partial} g_{i+1}^{(1)} + g_i^{(1)} e^{\alpha_i(\Phi)}) E_{i+1\,1},
$$

---

[4] For Δ integer or half-integer we have a well defined power (provided we make the choice of a point of order 2 in the Jacobian of Σ if Δ is strictly half integer) K^Δ and that a (meromorphic or C^∞) section of K^Δ can be identified with a collection of functions σ_α satisfying $\sigma_\alpha = k_{\alpha\beta}^{-\Delta}\sigma_\beta$ in each overlapping $U_\alpha \cap U_\beta$.

so that the first column reads

$$
\begin{aligned}
&\bar{\partial} g_2^{(1)} + e^{\alpha_1(\Phi)} \\
&\qquad \vdots \\
&\bar{\partial} g_n^{(1)} + g_{n-1}^{(1)} e^{\alpha_{n-1}(\Phi)}
\end{aligned}
\tag{4.5}
$$

As for the $(1,0)$ part of the connection we have

$$
\begin{aligned}
\left(D'\right)^{G_1} = {} & \sum_{i=1}^{n} \partial\varphi_i E_{ii} + g_2^{(1)} E_{11} + \mathcal{E}_+ \\
& + \sum_{j=1}^{n-2} \left(\partial g_{j+1}^{(1)} + g_{j+1}^{(1)} (\partial\varphi_{j+1} - \partial\varphi_1) - g_2^{(1)} g_{j+1}^{(1)} + g_{j+2}^{(1)} \right) E_{j+1\,1} \\
& + \left(\partial g_n^{(1)} + g_n^{(1)} (\partial\varphi_n - \partial\varphi_1) - g_2^{(1)} g_n^{(1)} \right) E_{n1} \, .
\end{aligned}
$$

We can at once infer that $g_2^{(1)} = -\partial\varphi_1$ and set to zero all the coefficients of $E_{j+1\,1}$, $j = 1,\ldots,n-2$ solving for $g_j^{(1)}$, $j = 3,\ldots,n$ together with the first column of $(D'')^{G_1}$, since, as was proven in [1], the Toda equations appear as the integrability condition for such a system.

Thus the first step essentially boils down to producing the "right" first column of the connection matrices. One can repeat *verbatim* the arguments above for the remaining columns. The consistency of the procedure relies in the fact that at step #k we kill all the elements of column k in the $(0,1)$-part, we create the term in the last row in the $(1,0)$ part while killing all others (except the one in row $k - 1$). It is not difficult to realize that such a configuration is left invariant by the subsequent gauge transformations $G_{k'}$ for $k' > k$.   □

The same procedure can be used to prove that the local gauge transformations G_i provide a C^∞ cochain that sends the cocycle $\mathcal{E}_{\alpha\beta}$ into the cocycle $\mathcal{V}_{\alpha\beta}$. It is interesting to notice that, in our framework, the derivatives $\partial\varphi_i$ of the Toda fields appear as (minus) the coefficients $g_{i+1}^{(i)}$ of the negative simple roots in the trivializing cochain. It is known [5] that the Toda equations globalize on a higher genus Riemann surface if the following non homogeneous gluing law holds:

$$
\phi_\alpha^i = \phi_\beta^i + \frac{1}{2} \sum_{l=0}^{i-1} (n - 2l - 1) \log |k_{\alpha\beta}|^2 \, .
\tag{4.6}
$$

Indeed, the transformation law between $\mathcal{E}_{\alpha\beta}$ and $\mathcal{V}_{\alpha\beta}$ reproduces the correct factors for the local fields $g_{\alpha\,i+1}^{(i)}$.

We now identify the holomorphic vector bundle defined by the Toda connection. This is accounted for by the following

Proposition 9. *Let V_n be a rank n flat vector bundle admitting a filtration*

$$V_1^{(n)} \subset V_2^{(n)} \subset \cdots \subset V_n^{(n)} \tag{4.7}$$

such that $V_{r+1}^{(n)}/V_r^{(n)} \simeq K^{(n-1)/2-r}$, $r = 1, \ldots, n-1$. If $g(\Sigma) \geq 2$, then V_n is the $(n-1)$th-jet extension of $K^{(n-1)/2}$.

Proof. Let us consider the sequence of quotients $\{K^{(n-1)/2}, K^{(n-1)/2-1}, \ldots, K^{-(n-1)/2}\}$. The last one gives the sequence

$$0 \to V_{n-1}^{(n)} \to V_n^{(n)} \to K^{-(n-1)/2} \to 0 , \tag{4.8}$$

giving $[V_n] \in H^1(K^{(n-1)/2} \otimes V_{n-1}^{(n)})$, where $[E]$ denotes the equivalence class of E. Tensoring with $K^{(n-1)/2}$ the sequences

$$0 \to V_r^{(n)} \to V_{r+1}^{(n)} \to K^{(n-1)/2-r} \to 0, \quad r = 1, \ldots, r-2 \tag{4.9}$$

and passing to the long cohomology sequences, we get the segments

$$H^1(V_r^{(n)} \otimes K^{(n-1)/2}) \to H^1(V_{r+1}^{(n)} \otimes K^{(n-1)/2}) \to H^1(K^{n-r-1}) \to 0. \tag{4.10}$$

But $V_1^{(n)} = K^{(n-1)/2}$ so that (4.10) gives $H^1(V_r^{(n)} \otimes K^{(n-1)/2}) = 0$ for $r = 1, \ldots, n-2$, and for $r = n-1$ we get the desired isomorphism

$$H^1(K^{(n-1)/2} \otimes V_{n-1}^{(n)}) \simeq H^1(K) \simeq \mathbb{C}. \tag{4.11}$$

The non-triviality of the extension follows from the fact that $V_2^{(n)} \subset V_n^{(n)}$ is $K^{(n-2)/2} \otimes V_2^{(2)}$ and that the cocycle defining the 1-jet extension of the spin bundle

$$0 \to K^{1/2} \to V_2^{(2)} \to K^{-1/2} \to 0 \tag{4.12}$$

is one half of the first Chern class of Σ.

By Weil's theorem, V_n cannot be decomposable into the direct sum of the powers of K appearing in the diagonals, and any two non-trivial extensions of F_1 by F_2 lying in the same ray in $H^1(\mathrm{Hom}(F_1, F_2))$ give rise to isomorphic vector bundles [31]. Thus V_n is seen to be isomorphic with the $(n-1)$-jet extension of $K^{(n-1)/2}$ once one notices that the latter has the same sequence of quotients as in the statement of the proposition. $\square$

Example: the sl(3) case

Let us examine in some detail the A_2 (*alias* $\mathfrak{sl}(3, \mathbb{C})$) case in order to clarify how the picture outlined above works.

The transition functions for the 2-jet bundle of K^{-1} (which is the case at hand) are given by the relations

$$\begin{pmatrix} \sigma_\alpha \\ \partial_\alpha \sigma_\alpha \\ \partial_\alpha^2 \sigma_\alpha \end{pmatrix} = \begin{pmatrix} k_{\alpha\beta} & 0 & 0 \\ \partial_\beta \log k_{\alpha\beta} & 1 & 0 \\ k_{\alpha\beta}^{-2} \partial_\beta^2 \log k_{\alpha\beta} & k_{\alpha\beta}^{-1} \partial_\beta \log k_{\alpha\beta} & k_{\alpha\beta}^{-1} \end{pmatrix} \begin{pmatrix} \sigma_\beta \\ \partial_\beta \sigma_\beta \\ \partial_\beta^2 \sigma_\beta \end{pmatrix} . \tag{4.13}$$

The smooth isomorphism between $V = J^2(K^{-1})$ and $E = K^{-1} \oplus \mathbb{C} \oplus K$ whose transition functions $\mathcal{E}_{\alpha\beta}$ are the diagonal part of Eq. (4.13), is accomplished by a *smooth*

$SL(3, \mathbb{C})$-valued 0-cochain G_α, which we seek in the factorized form (see the proof of Prop. 8) $G_\alpha = G_\alpha^{(1)} G_\alpha^{(2)}$ with

$$G_\alpha^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ h_\alpha & 1 & 0 \\ f_\alpha & 0 & 1 \end{pmatrix} , \qquad G_\alpha^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & g_\alpha & 1 \end{pmatrix} .$$

We will use the following standard overparametrization of the Toda field by means of the triple $[\varphi_1, \varphi_2, \varphi_3]$ related to the fields ϕ_1, ϕ_2 by $\phi_1 = \varphi_1 - \varphi_2$, $\phi_2 = \varphi_2 - \varphi_3$. Let us consider the Toda connection having the form

$$A' = \begin{pmatrix} \partial\varphi_1 & 1 & 0 \\ 0 & \partial\varphi_2 & 1 \\ 0 & 0 & \partial\varphi_1 \end{pmatrix} , \qquad A'' = \begin{pmatrix} 0 & 0 & 0 \\ e^{\varphi_1 - \varphi_2} & 0 & 0 \\ 0 & e^{\varphi_2 - \varphi_3} & 0 \end{pmatrix} . \qquad (4.14)$$

Under the transformation by $G_\alpha^{(1)}$ the cocycle (4.13) is sent into one of the form

$$\begin{pmatrix} k_{\alpha\beta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k_{\alpha\beta}^{-1}\partial_\beta \log k_{\alpha\beta} & k_{\alpha\beta}^{-1} \end{pmatrix}$$

provided we have in the overlappings

$$h_\alpha = k_{\alpha\beta}^{-1} h_\beta - k_{\alpha\beta}^{-1}\partial_\beta \log k_{\alpha\beta} ,$$

and

$$f_\alpha = k_{\alpha\beta}^{-2} f_\beta + k_{\alpha\beta}^{-1} h_\beta \partial_\beta \log k_{\alpha\beta} - k_{\alpha\beta}^{-3}\partial_\beta^2 \log k_{\alpha\beta} .$$

It is then easy to see that the reduction of the cocyle (4.13) to its diagonal part is accomplished by the transformations $G_\alpha^{(2)}$ provided $g_\alpha = k_{\alpha\beta}^{-1} g_\beta - k_{\alpha\beta}^{-1}\partial_\beta \log k_{\alpha\beta}$.

More interesting is the transformation of the connection (4.14), which we display below dropping the indices referring to the coordinate patches:

$$(A')^{G^{(1)}} = \begin{pmatrix} \partial\varphi_1 + h & 1 & 0 \\ \partial h + h(\partial\varphi_2 - \partial\varphi_1) + f - h^2 & \partial\varphi_2 - h & 1 \\ \partial f + f(\partial\varphi_3 - \partial\varphi_1) - hf & -f & 0 \end{pmatrix} ,$$

$$(A'')^{G^{(1)}} = \begin{pmatrix} 0 & 0 & 0 \\ e^{\varphi_1 - \varphi_2} + \bar\partial h & 0 & 0 \\ h\, e^{\varphi_2 - \varphi_3} + \bar\partial f & e^{\varphi_2 - \varphi_3} & 0 \end{pmatrix} .$$

This means that we have to solve for the equations

$$\partial\varphi_1 + h = 0 , \qquad \partial h + h(\partial\varphi_2 - \partial\varphi_1) + f - h^2 = 0 ,$$

$$e^{\varphi_1 - \varphi_2} + \bar\partial h = 0 , \qquad h\, e^{\varphi_2 - \varphi_3} + \bar\partial f = 0 ,$$

which gives $h = -\partial\varphi_1$ and $f = \partial^2\varphi_1 + \partial\varphi_1\partial\varphi_2$, the other two equations being identically true on the solutions of the Toda equations. The action of the subsequent gauge transformation $G^{(2)}$ gives $g = \partial\varphi_3$ and sends the Toda connection into the form

$$(A')^G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ w_3 & w_2 & 0 \end{pmatrix}, \qquad (A'')^G = 0, \tag{4.15}$$

where w_2 is the usual energy momentum tensor of the sl_3 Toda theory,

$$w_2 = (\partial\phi_1)^2 + (\partial\phi_2)^2 - [\partial^2\phi_1 + \partial^2\phi_2 + \partial\phi_1\partial\phi_2], \tag{4.16}$$

and

$$w_3 = \partial w_2 + (\partial\phi_1)^2\partial\phi_2 - \partial\phi_1(\partial\phi_2)^2 + 2\partial\phi_2\partial^2\phi_2 - \partial^3\phi_2 \tag{4.17}$$

is the other generator of the W_3-algebra.

5. Embeddings and *W*-geometry

The purpose of this section is to discuss more thoroughly the geometrical features of (standard) Toda Field Theory. In particular, we shall discuss the embeddings of a Riemann surface determined by the classifying map resulting from the self-duality equations, see Section 2. The crucial properties for this analysis are the natural filtration of the Higgs bundle, and the fact that, at least in the standard Toda case, there is an additional real structure preserved by the Toda connection. As in Section 4, our discussion will be confined to the A_n case.

5.1. The additional real structure

Consider the basic Higgs system given by (3.7) and (2.8) together with the harmonic metric H. By the general discussion in Section 2 we know that the structure group of E as a harmonic bundle reduces to $SL(n, \mathbb{R})$. We now show that there is another real structure compatible with this one. Let $A : E \to E$ be the endomorphism equal to $(-1)^r$ on each factor $K^{-(n-1)/2+r}$. With it, we construct an indefinite hermitean form $\langle \cdot, \cdot \rangle$ over E, namely

$$\langle u, v \rangle = (A u, v)_H, \quad u, v \in E. \tag{5.1}$$

A straightforward calculation proves

Lemma 10. *The hermitean form (5.1) is flat with respect to the Toda connection* $D = D_H + \theta + \theta^*$, *that is we have*

$$\mathrm{d}\langle u, v \rangle = \langle D u, v \rangle + \langle u, D v \rangle, \quad u, v \in E.$$

This implies, of course, that a reduction of the structure group from $SL(n, \mathbb{C})$ to $SU(p, q)$, where $p = [n/2]$, $q = n - p$, takes place. More precisely, what we actually mean by "$SU(p, q)$" is the group corresponding to the fixed point set in $\mathfrak{g}^{\mathbb{C}} = A_{n-1}$ of the conjugation ν given by

$$\nu(\xi) = -I\, \rho(\xi)\, I\,, \quad \xi \in \mathfrak{g}^{\mathbb{C}}\,, \tag{5.2}$$

where in this case ρ is simply minus the hermitean conjugate and I is the diagonal matrix

$$I = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix} \tag{5.3}$$

(the sign in the last element being determined according to the parity of n). It is obvious that this is the standard form for $SU(p, q)$ up to a coordinate reshuffling. It is also easy to see that the conjugations τ defined in Section 2 (Eq. (2.11)) and ν commute, so that (the Lie algebra of) the structure group of the harmonic bundle corresponding to the Toda equations is in fact given by the intersection of the fixed point sets of τ and ν. Let us call G the real structure group so obtained. We further define K to be its maximal compact subgroup.

By the results about harmonic bundles quoted in Section 2, we thus obtain a harmonic map

$$f_H : \widetilde{\Sigma} \longrightarrow G/K\,,$$

where $\widetilde{\Sigma}$ is the universal cover of Σ, or, in other terms, a map

$$f_H : \Sigma \longrightarrow \Gamma \backslash G/K\,,$$

where the discrete subgroup $\Gamma \subset G$ is the image of $\pi_1(\Sigma)$ through the holonomy. By a little abuse of language, we use the same symbol for both. For the sake of convenience, let us denote $\widetilde{N} = G/K$ and $N = \Gamma \backslash G/K$. The stability properties of the Higgs bundle we are looking at [28] imply the action of Γ on $\widetilde{N}$ to be properly discontinuos, so that N is a manifold.

Thus we can interpret the Toda field equations as the equations characterizing the embedding of the Riemann surface into a some homogeneous manifold N through a harmonic map f_H. We can actually refine this, that is, starting from the map f_H or – what is the same – from the harmonic bundle we can construct an embedding $F : \Sigma \to \mathbb{D}$ into a *complex* manifold $\mathbb{D}$. This requires analyzing more extensively the structure of the bundle we associated to the Toda equations.

5.2. Toda systems and variations of Hodge structures

Upon rewriting our rank-n basic bundle (2.8) as [13]

$$E = \bigoplus_{r+s=n-1} E^{r,s}, \qquad E^{r,s} = K^{-(n-1)/2+r}, \tag{5.4}$$

the Higgs field θ appearing in the Toda connection, Eq. (3.7), has the property

$$\theta : E^{r,s} \longrightarrow E^{r-1,s+1} \otimes K \tag{5.5}$$

and the factors are orthogonal with respect to both the metric H and the indefinite hermitean form $\langle \cdot, \cdot \rangle$. As a consequence, the complete connection $D = D_H + \theta + \theta^*$ satisfies the following *Griffiths transversality condition*:

$$D : E^{r,s} \longrightarrow A^{1,0}(E^{r-1,s+1}) \oplus A^{1,0}(E^{r,s}) \oplus A^{0,1}(E^{r,s}) \oplus A^{0,1}(E^{r+1,s-1}), \tag{5.6}$$

where by $A^\bullet(E^{r,s})$ we mean C^∞ sections. It is useful for later purposes to rewrite (5.6) in the following form. Consider the filtration

$$E \equiv F^0 \supset F^1 \supset \cdots \supset F^{n-1} \supset F^n \equiv \{0\},$$

where

$$F^q = \bigoplus_{r=q}^{n-1} K^{-(n-1)/2+r} = \bigoplus_{r=q}^{n-1} E^{r,s}. \tag{5.7}$$

Then the transversality condition can be restated as

$$D' : F^q \longrightarrow A^{1,0}(F^{q-1}), \qquad D'' : F^q \longrightarrow A^{0,1}(F^q). \tag{5.8}$$

Notice that $\operatorname{rk} F^q = n - q$ and that these are the subbundles[5] corresponding to the filtration of V by the $V_q^{(n)}$'s appearing in Section 4. In the purely holomorphic picture (5.8) reads

$$\nabla' : V_q^{(n)} \longrightarrow \Omega^1(V_{q-1}^{(n)}), \tag{5.9}$$

where $\Omega^1(\cdot)$ denotes the space of holomorphic differentials.

According to Simpson, a harmonic bundle $E = \bigoplus_{r+s=w} E^{r,s}$ whose factors are orthogonal with respect to an indefinite hermitean form $\langle \cdot, \cdot \rangle$, satisfying any one of (5.5)–(5.9) defines a *complex variation of Hodge structure* [33,34,23][6] of weight w.

Thus the Higgs bundle associated to the Toda equations displays the formal properties of a Variation of Hodge Structure of weight $w = n - 1$, whose "Hodge Bundles" $E^{r,s} = K^{-(n-1)/2+r} \cong F^r/F^{r+1}$ are in fact line bundles. We shall use the machinery of Variations of Hodge Structure to prove

Theorem 11. *The Toda equations determine a holomorphic embedding*

$$F_H : \Sigma \longrightarrow \Gamma \backslash \mathbb{D},$$

---

[5] Up to a reshuffling of indices.

[6] The main difference from Griffiths' original definition is that in Simpson's the existence of the integral lattice is left out. We shall stick to this one.

where Γ is the monodromy group, $\mathbb{D} \cong G/K_0$ a Griffiths period domain, G is the structure group defined in Section 5.1 and $K_0 \subset K \subset G$ a (compact) subgroup. The map F_H is the metric H seen as a section of a flat bundle over Σ with typical fibre G/K_0 and its differential is given by the Higgs field θ.

For the proof we need to recall some basic properties of Griffiths period domains (or classifying spaces).

A brief tour through period domains

Let us denote by $\mathbf{E}$ a complex vector space equipped with
- a conjugation $\cdot^\sigma : \mathbf{E} \to \mathbf{E}$,
- a bilinear form $Q : \mathbf{E} \times \mathbf{E} \to \mathbb{C}$ such that:
    (i) $Q(v, u) = (-1)^w Q(u, v)$, $u, v \in \mathbf{E}$,
    (ii) it is "real" with respect to the conjugation of $\mathbf{E}$, namely $\overline{Q(u,v)} = Q(u^\sigma, v^\sigma)$, $u, v \in \mathbf{E}$.

A period domain $\mathbb{D}$ is the set of all weight w Hodge structures on $\mathbf{E}$, namely the set of all decompositions $\mathbf{E} = \bigoplus_{r+s=w} \mathbf{E}^{r,s}$ satisfying

$$Q(\mathbf{E}^{r,s}, \mathbf{E}^{r',s'}) = 0 \text{ unless } r' = s \text{ and } s' = r,$$
$$i^{r-s} Q(u, u^\sigma) > 0 \text{ for any } u \in \mathbf{E}^{r,s}.$$

If the weight w is $n - 1$, which is the case we will be dealing with, then all the factors in a given Hodge structure are actually lines, but this is not quite so in general. It is useful to define the same object in terms of filtrations. To this end, define the operator $C : \mathbf{E} \to \mathbf{E}$ by $C|_{\mathbf{E}^{r,s}} = i^{r-s}$. Then $\mathbb{D}$ is defined to be the set of all (descending) filtrations $\{\mathbf{F}^q\}$ in $\mathbf{E}$ such that

$$Q(\mathbf{F}^q, \mathbf{F}^{w-q+1}) = 0, \qquad Q(Cu, u^\sigma) > 0. \tag{5.10}$$

The link between the two definitions lies in the decomposition $\mathbf{F}^q = \bigoplus_{r=q}^{w} \mathbf{E}^{r,s}$. Dropping the second condition in either one of the two definitions we just gave, yields the *compact dual* $\check{\mathbb{D}}$ of $\mathbb{D}$. It is an algebraic subvariety of a flag manifold, and hence of a product of Grassmannians [24]. This is clear from the second definition. This implies for $T_{\check{\mathbb{D}}}$, the holomorphic tangent bundle of $\check{\mathbb{D}}$, that

$$T_{\check{\mathbb{D}}} \subset \bigoplus_{q=1}^{w} \mathrm{Hom}(\mathbf{F}^q, \mathbf{E}/\mathbf{F}^q) = \bigoplus_{q=1}^{w} \bigoplus_{r=1}^{q} \mathrm{Hom}(\mathbf{E}^{q,w-q}, \mathbf{E}^{q-r,w-q+r}).$$

The group $G^{\mathbb{C}} = \mathrm{Aut}(\mathbf{E}, Q)$ acts transitively on it [23], thus $\check{\mathbb{D}}$ is in fact a complex *manifold*. The period domain $\mathbb{D}$ lies inside it as an open subset, and therefore as a complex submanifold. It is the open orbit through the origin of $\check{\mathbb{D}}$ of the real form of $G_\mathbb{C}$ with respect to the given conjugation in $\mathbf{E}$.

Proof of Theorem 11

We start with

Lemma 12. *The group G introduced in subsection 5.1 is the real group acting on the classifying space of weight $n - 1$ Hodge structures on a vector space of dimension n.*

Proof. The choice of a basis $e_1, \ldots, e_n$ in a complex vector field $\mathbf{E}$ of dimension n allows to define the decomposition

$$\mathbf{E} = \bigoplus_{r+s=n-1} \mathbf{E}^{r,s}, \quad \mathbf{E}^{r,s} = \mathbb{C}\{e_{r+1}\},$$

and an indefinite hermitean form through

$$\langle e_i, e_j \rangle = \delta_{ij}(-1)^{i+1}.$$

Then the form $\langle \cdot, \cdot \rangle$ is the one represented by the matrix I of (5.3). We can moreover consider the conjugation map

$$\cdot^\sigma : \mathbf{E} \longrightarrow \mathbf{E}, \quad u \mapsto u^\sigma = S\bar{u},$$

where S is the matrix (2.10). Notice that with this definition we have $(\mathbf{E}^{r,s})^\sigma = \mathbf{E}^{s,r}$. Thus the above decomposition is a (reference) Hodge structure in $\mathbf{E}$ of weight $n - 1$ [23,24]. Then we define the bilinear form Q by

$$Q(u, v) = i^{n-1}\langle u, v^\sigma \rangle, \quad u, v \in \mathbf{E}.$$

It is easy to check that it has the properties listed before in the résumé of period domains. We finally introduce the complex Lie group $G_\mathbb{C} = SO(Q, \mathbb{C}) \equiv \mathrm{Aut}(\mathbf{E}, Q)$ of those complex automorphisms of $\mathbf{E}$ which preserve Q.

Recall now that G has been defined as the real group arising as intersection of the fixed point sets of the conjugations τ and ν in (2.11), (5.2). Therefore it preserves the hermitean form $\langle \cdot, \cdot \rangle$. It is then obvious that G coincides with the real subgroup of $G_\mathbb{C}$ defined by the conjugation $\cdot^\sigma$, as we clearly have $(g u)^\sigma = \tau(g) u^\sigma$, for $u \in \mathbf{E}$ and $g \in G_\mathbb{C}$.

The statement now follows from the fact that the period domain $\mathbb{D}$ is the quotient G/K_0 [23,24], where $K_0 = G \cap B$ and $B \subset G_\mathbb{C}$ is the stabilizer of the reference Hodge structure. □

Remark 13. The stabilizer group K_0 is in general *strictly* contained in the maximal compact subgroup K of G.

Let us now come back to the Higgs bundle E equipped with the filtration $\{F^q\}$ defined in Eq. (5.7).

To construct the mapping $F_H : \Sigma \to \Gamma \backslash \mathbb{D}$, choose a basepoint x_0 on Σ. Then we look at the fibre E_{x_0} as the fixed vector space $\mathbf{E}$. Notice that the conjugation in the proof of the preceding lemma agrees with the one in E constructed in Section 2. Thus we repeat the constructions in the proof of Lemma 12 and get the hermitean form $\langle \cdot, \cdot \rangle_{x_0}$ and the required bilinear form Q as well. Since the connection D is flat, we can use the isomorphism of any fibre E_x with E_{x_0} to induce a filtration on E_{x_0} (which will be the

image of the filtration $\{F_x^q\}$ on E_x). The hermitean form $\langle \cdot, \cdot \rangle$ on E is *flat* under D, thus the orthogonality properties of the subspaces in the fibre E_x translate into the bilinear relations for the induced filtration on E_{x_0}. Therefore the reference Hodge structure in E_x defines a new Hodge structure in E_{x_0} and we obtain a local map from Σ to $\mathbb{D}$. The entire construction is of course defined up to the action of the monodromy group.

Lemma 14. *The mapping* $F_H : \Sigma \to \Gamma \setminus \mathbb{D}$ *so defined is holomorphic.*

Proof. The statement is local. By construction, the differential of F_H is defined by the flat connection D itself. From the inclusion $\mathbb{D} \subset \check{\mathbb{D}}$ we have [24]

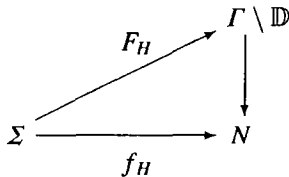$$T_\mathbb{D} \subset \bigoplus_{q=1}^{n-1} \mathrm{Hom}(\mathbf{F}^q, \mathbf{E}/\mathbf{F}^q)$$

and using this picture for the tangent bundle, holomorphicity follows from the transversality condition $D'' : F^q \to A^{0,1}(F^q)$. □

Remark 15. According to the given description of $T_{\check{\mathbb{D}}}$, the other half of the transversality condition says that $F_{H*}(T_\Sigma) \subset \bigoplus_{r=1}^{n-1} \mathrm{Hom}(\mathbf{E}^{r,n-1-r}, \mathbf{E}^{r-1,n-r})$. This is the geometrical meaning of the *grading condition* imposed on the Toda connection [21,32].

Finally, we can conclude that the map F_H is essentially the metric H, seen as a section of a bundle in homogeneous spaces, by applying the same argument we used in Section 2. The key point is to notice that the Variation of Hodge Structure defines a reduction of the structure group G to its intersection with the stabilizer of the reference Hodge structure (the group we called K_0 before) [33], and it is obvious that the section so obtained coincides with the metric H. The theorem is proved. □

Remark 16. We notice that the targets of the holomorphic embeddings so obtained *depend on the genus* of the Riemann Surface, see Section 6 below. This is a consequence of the holonomy representation of the fundamental group on the Griffiths period domain $\mathbb{D}$.

The situation with our embeddings is the following. Using the fibration $\mathbb{D} \cong G/K_0 \to G/K \equiv \tilde{N}$ [23] we have the diagram

$$
\begin{array}{ccc}
 & & \Gamma \setminus \mathbb{D} \\
 & \nearrow^{F_H} & \downarrow \\
\Sigma & \xrightarrow[f_H]{} & N
\end{array}
$$

where the map f_H is harmonic (N is in general not complex) and F_H is holomorphic. This is an instance of a more general situation in which harmonic maps are covered by Variations of Hodge Structure [10].

The complete classification of the homogeneous spaces for the A_n case has been performed by Griffiths [23], as

$$\mathbb{D} = \begin{cases} SO(n+1, n, \mathbb{R})/U(1)^n & \text{for } A_{2n}, \\ Sp(2n, \mathbb{R})/U(1)^n & \text{for } A_{2n-1}. \end{cases}$$

Our favourite example of A_2 can be easily worked out completely. In this case we Q is represented by the matrix

$$\begin{pmatrix} & & -1 \\ & 1 & \\ -1 & & \end{pmatrix}. \tag{5.11}$$

Looking at the left half of the Hodge diamond, the first bilinear relation (5.10) yields the divisor $2X_0X_2 - X_1^2 = 0$ in $\mathbb{P}^2$, the rational normal curve given by the Veronese embedding, while the second selects the complement of the real circle $|z|^2 = 2$ in this rational curve, i.e., we have the disjoint union of two copies of the Poincaré disk.

We shall conclude this section with a brief comment on the Conformal Affine case. Most of the structure described in this section does not carry over directly to this more general case. In particular, Lemma 10 is easily seen to be false if θ is given by (3.9). Therefore the second real structure cannot be defined by means of the endomorphism I, which implies that we cannot follow the path of the standard Toda case to define a Variation of Hodge Structure and we cannot use the holomorphic embedding into the Griffiths period domain any longer.

A possible way out can be conceived along the following lines. The real structure described in Section 2 is not ruled out by the deformation leading to the Conformal Affine Toda system and therefore there is still a map

$$f_H : \widetilde{\Sigma} \longrightarrow SL(n, \mathbb{R})/SO(n).$$

The target manifolds not being complex, the map f_H above is not suitable as it stands to construct holomorphic embeddings. Anyhow, since the map f_H is harmonic and Σ has complex dimension 1, we will have

$$\widetilde{\nabla}'' \partial f_H = 0, \tag{5.12}$$

where in this case ∂f_H is to be understood as a section of the vector bundle

$$T_\Sigma^* \otimes f_H^*(T_{SL(n,\mathbb{R})/SO(n)}^{\mathbb{C}})$$

and $\widetilde{\nabla}$ is the tensor product of the Kähler connection on $K \equiv T_\Sigma^*$ and the pull-back of the Riemannian one on $T_{SL(n,\mathbb{R})/SO(n)}^{\mathbb{C}}$, respectively [10]. Thus ∂f_H is a holomorphic section of a certain complex vector bundle over Σ. Since the map f_H is determined by the metric H, by Eq. (2.7) we have that Eq. (5.12) is the translation in this framework of $\bar{\partial}\theta = 0$.

6. Conclusions and comments

In this paper we have analyzed the "triangular" correspondence [21,26]

$$\text{Toda} \leftrightarrow W_n - \text{algebras} \leftrightarrow \text{Higgs bundles}$$

from the point of view of the theory of hermitean holomorphic vector bundles on a generic genus Riemann surface Σ. Although the origin of such relationships can be traced back to the fact that both the Toda Field and Hitchin's self-duality equations are dimensional reductions of a suitable four-dimensional Self-Dual Yang–Mills theory, we deemed it worthwhile to work out some topics from the two-dimensional viewpoint.

In particular we have proved that the assignement of a solution of the A_{n-1} Toda Field equations determines both in standard and in the Conformal Affine case a harmonic Higgs bundle. The metric is parameterized by the Toda fields, and the Higgs field arises as the non metric part of the total flat connection.

The underlying holomorphic vector bundle V is uniquely fixed to be the bundle of $(n-1)$th-jets of sections of $K^{-(n-1)/2}$. W_n fields are naturally identified with the non trivial entries of the flat *analytic* connection on V. Actually, this is not an unexpected feature and it has already been found out e.g. in [9]; however, we would like to point out that a very nice geometrical significance of the Toda fields as the building blocks of the local isomorphisms (in the C^∞-category) between the two holomorphically distinct bundles $E = \bigoplus_{r=0}^{n-1} K^{-(n-1)/2+r}$ and $V = J(K^{-(n-1)/2})$ is clarified together with some global features of the higher genus case which were perhaps a bit overlooked in the literature.

The main point can be considered the discussion of how the datum of a A_{n-1}-Toda Field on Σ leads, in the standard case, to the realization of the Riemann Surface as a base space for a Variation of Hodge Structure of weight $n-1$ and rank n, and henceforth a holomorphic map from Σ into a quotient of a Griffiths period domain G/K_0.

Since these results go in the direction of the so called geometry of W_n-embeddings as put forward by Gervais, Saveliev and collaborators, a couple of comments are in order.

First of all, in the paper [32] the following picture is explained. The starting point is a C^∞-map from Σ to a complex Lie group G which, under suitable instances (the "grading condition") lifts a holomorphic map $\varphi_P : \Sigma \to G/P$, P being a parabolic subgroup of G. Considering those parabolic subgroups $P_i, i = 1, \ldots, \text{rank } G$, for which G/P_i is the ith fundamental homogeneous space for G, the associated maps φ_{P_i} define maps from Σ to $\mathbb{P}(V_i)$, the projectivisation of the ith fundamental representation of G. Then it is shown that the (generalized) Plücker relations for the curvature of the pull-back on Σ of the Fubini–Study metrics on $\mathbb{P}(V_i)$ on Σ translate, when expressed through local Kähler potentials, into the Toda Field equations for a suitably chosen local representative of φ_{P_i}.

Our starting point is different: we *start* from a solution of the Toda Field equations and we *determine* a holomorphic map from Σ to a suitable locally homogeneous space. It follows that the target space we obtain is only *locally* determined by the rank of the Cartan subalgebra in which the Toda fields take values, since in the large the monodromy

action of the fundamental group of Σ on the Griffiths period domain must be factored out, thus yielding a different global target space according to the genus $g(\Sigma)$.

Nonetheless, Plücker formulas are of local type, so one should expect them to arise also in our context. Indeed, one can see that they can be recovered by considering the natural embeddings of the algebraic manifold $\check{\mathbb{D}}$ into the product of Grassmannians [24]

$$G(h_1, n) \times G(h_2, n) \cdots \times G(h_{n-1}, n)$$

(h_r is the rank of F^r in the Hodge filtration) and pulling back to $\check{\mathbb{D}}$ the determinant line bundles associated to the tautological sequences

$$0 \longrightarrow S_{h_r} \longrightarrow \mathbb{C}^n \longrightarrow Q_{h_r} \longrightarrow 0$$

over $G(h_r, n)$. Plücker coordinates for $G(h_r, n)$ are indeed obtained by taking holomorphic sections of $\det Q_{h_r}$. It is to be borne in mind that $\check{\mathbb{D}}$ is *strictly* contained in the complete flag manifold for $SL(n, \mathbb{C})$: the A_{n-1}-type Plücker formulas ensuing from the various tautological sequences can be obtained by explicitly realizing the embedding of $\check{\mathbb{D}}$ in the product considered above.

The following, and final, remark is also partly motivated by the last observation. We have shown that the Toda connection D is compatible with the two Lie algebra automorphisms (2.11) and (5.2) which we rewrite here:

$$\tau(X) = S\bar{X}S, \qquad \nu(X) = -I\bar{X}^t I.$$

Following [17], we notice that $\hat{\tau} = \tau\nu = \nu\tau$ is the extension of the automorphism of the A-type Dynkin diagram which can be used to define the algebras B_r and C_r as quotient of A_{2r} and A_{2r-1}, respectively. Moreover, since the Griffiths period domains in which $\tilde{\Sigma}$ is immersed are quotients of $SO(r+1, r)$ and $Sp(2r, \mathbb{R})$, respectively, we are led to conclude that the maps induced by the metric might give insight into the geometry of the $W(B)$ and $W(C)$-algebras obtained via the folding procedure by the original A-theory.

We hope to discuss those questions more thoroughly in a future work.

Acknowledgement

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